# FREQUENCY SPECTRUM OF THE LATERAL FIELD EXCITED CONVEX PIEZOELECTRIC RESONATOR OF THICKNESS-SHEAR VIBRATIONS

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## ABSTRACT

The mathematical model is discussed of the lateral field excited crystal resonator employing thickness-shear vibrations of the even anharmonic mode. The resonator is combined with a convex piezoelectric plate and two one side placed electrodes of the relevant shape. It is shown that the electrode shape causes structure of the vibration frequency spectrum, which differs from the traditionally calculated for the symmetrically placed electrodes. Relevant approximate formula for the frequency spectrum is obtained via the Airy functions. Influence of the electrode positioning on the resonator surface is also shown and discussed.

## 1. INTRODUCTION

The recently proposed modulation method [1] allows getting progress in self-contained frequency control of crystal oscillators owing to use of the resonator anharmonic modes as sensors of environment. The approach requires accurate determination of the resonator frequency spectrum and sensor frequencies to avoid the problems with the modes interaction and intermodulation. Modulation may cause, in principle, the nonlinear products, which fall within a bandwidth of the sensor, and, as a result, both the unwanted splashes in the phase spectrum of oscillator and irregularities of the sensitivity curve of a sensor may appear. Either phenomenon can readily be prevented at the early stage if the electrode shapes and piezoelectric plate geometry are slightly adjusted based on a proper mathematical model of a resonator. To obtain, the mathematical model of a resonator, first, must be adjusted for the lateral field based design and, then, studied for the frequency spectrum.

Traditionally, calculation of the crystal resonator frequency spectrum is carried out mostly for two side placed electrodes. Of course, such an approach based on a solution of either linear [2] or nonlinear vibration equations of a piezoelectric resonator and developed in many papers gives practical accuracy for many applications, include environment sensitivity (temperature, acceleration, etc.). Even so, for the sake of accuracy, the resonator mathematical model requires to be improved for the one side placed electrodes of an arbitrary shape if one starts to solve the aforementioned task of an accurate calculation of the resonator frequency spectrum.

This report addresses the mathematical model of a resonator with a convex piezoelectric plate with one side placed electrodes employing thickness-shear vibrations. Intentionally, we relate the results to the lateral field excited crystal resonator employing the nearest even anharmonic mode as a fundamental vibration [3]. Such a resonator design is not yet practically studied for the possible anharmonic sensors of environment. Even so, the fact that a resonator exhibits a one electrodeless free surface makes it to be very attractive for the extra low noise units of the precision family. Having such a potential, certainly, it should be peer examined. Before starting with the mathematical routine and necessary explanations, it is just in order to show that, owing to the one sided electrodes, its frequency spectrum has a certain structure, obeying the approximate nonlinear relation

$$\omega_{nkl}^2 \cong \widehat{\omega}^2 + P(k - 1/4)^{2/3} + Ql, \qquad (1)$$

where, traditionally, $n$ is a number of overtone, both integer $k$ and $l$ characterize index of an anharmonic mode; $n, k, l = 1, 2, \ldots$; $\widehat{\omega}$, $P$ and $Q$ are constants determined by solution of a differential vibration equation of a crystal resonator; $\omega_{nkl}$ is an arbitrary angular vibration frequency. Even a quick look at the formula (1) shows that it differs from that obtained for the symmetrically placed two side electrodes and discussed in detail in many papers, in [2], [4], and [5], in particularly.

## 2. QUASI ONE-DIMENSIONAL FREQUENCY SPECTRUM

Now let us examine the aforementioned crystal resonator design (Fig. 1) with a circular convex piezoelectric plate, which convex radius of an upper surface is $R$ and which maximum thickness is $L$. Here two electrodes of the proper shape are placed symmetrically on the upper convex surface. If to follow [4] and [6], then the vibration frequency spectrum of a resonator may be found by a solution of the effective differential equation in the self axes x and y of the tensor $d_{\alpha\beta}$

$$-d_1\frac{\partial^2\Psi}{\partial x^2}-d_2\frac{\partial^2\Psi}{\partial y^2}+\omega_n^2\left(1+\frac{x^2+y^2}{RL}\right)\Psi(x,y)=\omega^2\Psi(x,y) \tag{2}$$

where $d_1$ and $d_2$ are eigenvalues of a tensor $d_{\alpha\beta}$ coupled with the piezoelectric constants [4] and [6], and $\omega_n$ is a fundamental frequency of the $n$ overtone.

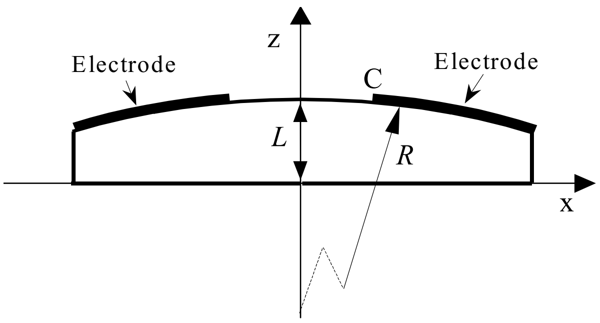


**Figure 1. Model of a lateral field excited crystal resonator with a convex piezoelectric plate (intersection): x and z are rotated axes of the crystal plate.**

In the considered frequency range

$$\omega_{nkl}^2-\omega_n^2<<\Delta\,, \tag{3}$$

where $\Delta=\omega_{nF}^2-\omega_n^2$ is determined by an abrupt change in a potential at the electrode bound $\Gamma$, $\omega_{nF}$ and $\omega_n$ are, respectively, the fundamental frequencies beyond and within the electrode. Zero condition [7] may be used at the electrode bound $\Gamma$ if to assume $\Delta=\infty$

$$\Psi(x,y)\big|_{\Gamma}=0\,, \tag{4}$$

which corresponds to the first order of accuracy for the small parameter $\frac{\omega_{nkl}^2-\omega_n^2}{\Delta}<<1$. If $\Delta$ is assumed to be limited then a solution is seeking for the higher orders of accuracy over the mentioned small parameter.

Translating (2) to the dimensionless variables yields

$$\begin{cases}\xi=\dfrac{1}{\lambda_n}(x\cos\alpha-y\sin\alpha)\\[2mm] \eta=\dfrac{1}{\lambda_n\sqrt{\varepsilon}}(x\varepsilon\sin\alpha+y\cos\alpha)\end{cases}, \tag{5}$$

the new equation appears in a form of

$$\frac{\partial^2\Psi}{\partial\xi^2}+\frac{\partial^2\Psi}{\partial\eta^2}+\Big[\theta-U_0(\xi)-U_1(\xi)[\eta-\varphi(\xi)]+\frac{\varepsilon}{\gamma_0^3}[\eta-\varphi(\xi)]^2\Big]\Psi=0 \tag{6}$$

with the same zero condition at the bound $\Gamma$, achieved if to assume that $\eta=\varphi(\xi)$. Here $\varepsilon=d_2/d_1$, $\gamma_0=\cos^2\alpha+\varepsilon\sin^2\alpha$, $\omega^2=\omega_n^2+\gamma_0 d_1\theta/\lambda_n^2$, and $\lambda_n=(d_1RL/\omega_n^2)^{1/4}$.

Rigorous forms of $U_0(\xi)$ and $U_1(\xi)$ for (6) appear straightforward from (2) if to account the new variables (5), these are

$$U_0(\xi)=\frac{1}{\gamma_0^3}[\varepsilon\varphi^2(\xi)+\sqrt{\varepsilon}(1-\varepsilon)\sin(2\alpha)\xi\varphi(\xi)+(1-(1-\varepsilon^2)\sin^2\alpha)\xi^2],$$
$$U_1(\xi)=\frac{1}{\gamma_0^3}\big[2\varepsilon\varphi(\xi)+\sqrt{\varepsilon}(1-\varepsilon)\sin(2\alpha)\xi\big]. \tag{7}$$

If now to approximate the electrode bound with a circle of a radius $a$ around the point close to the center of a piezoelectric plate (Fig. 2) then a function $\varphi(\xi)$ may readily be found in a form of

$$\varphi(\xi)=-\frac{1-\varepsilon}{2\sqrt{\varepsilon}}\sin(2\alpha)\xi+\frac{\gamma_0}{\sqrt{\varepsilon}}\left(\frac{a+b}{\lambda_n}-\sqrt{\left(\frac{a}{\lambda_n}\right)^2-\xi^2}\right). \tag{8}$$

We have now enough to solve (6) for the required function $\Psi(\xi,\eta)$. The proper solution with account of (8) is then appears in a form of

$$\Psi(\xi,\eta)=U_1^{1/6}(\xi)\sum_k f_k(\xi)\mathbf{Ai}(U_1^{1/3}(\xi)(\eta-\varphi(\xi))-s_k), \tag{9}$$

where $\mathbf{Ai}(x)$ is Airy function, which roots are negative and equal to $-s_k$. It may easily be shown that function (9) equals zero with $\eta=\varphi(\xi)$ that, in turn, means that the achieved solution satisfies the boundary condition (4). Furthermore, substituting (9) for (6) yields the differential equations system for $f_k(\xi)$, which may be solved, for example, with a series approach for the small parameter $\frac{\lambda_n}{a}<<1$, allowing decomposition of the equation coefficients for the orders of $\xi$. Such a manipulation with neglecting of the small terms produces the relation for the frequencies $\omega_{nkl}$ of a resonator spectrum in a form of

$$\omega_{nkl}^2 = \omega_n^2 \left\{ 1 + \frac{b^2}{RL} \left[ \frac{1}{\gamma_0^2} + \left( \frac{\lambda_n}{b} \right)^{4/3} \beta s_k + 2 \left( \frac{\lambda_n}{b} \right)^2 \sqrt{BM_k} \left( l - \frac{1}{2} \right) \right] \right\}, \quad (10)$$

where $n, k, l = 1,2,...$ ,

$$B = 1 + \frac{1}{\gamma_0} \left( \frac{b}{a} - \frac{(1-\varepsilon)^2}{2} \sin^2(2\alpha) \right),$$
$$M_k = 1 - 4 \left( 1 - \frac{1}{1 + \left( \frac{1-\varepsilon}{2\sqrt{\varepsilon}} \sin 2\alpha \right)^2} \right) \mu_k, \quad (11)$$

and $b$ is a minimum distance between a center of a piezoelectric plate and the electrode bound of the convex radius $a$, $\alpha$ is an angle between an electrode axis and a self axis y of a tensor $d_{\alpha\beta}$ (Fig. 2).

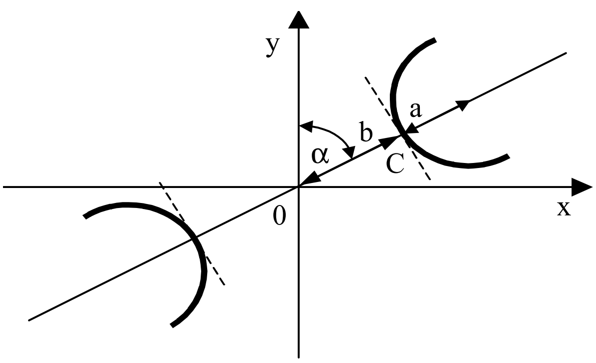


**Figure 2. One side placement of the electrodes relatively the self axes of the tensor $d_{\alpha\beta}$**

Furthermore, the parameter $\beta$ in (10) is given by

$$\beta = \frac{1}{\gamma_0^{1/3}} [4\varepsilon + (1-\varepsilon)^2 \sin^2(2\alpha)]. \quad (12)$$

It is now in order to note that, having a common sense, the numbers $\mu_k$ in (11) do not depend on the electrode size and geometry, and are calculated by

$$\mu_k = \frac{1}{(\mathbf{Ai}'(-s_k))^2} \sum_{l \neq k} \frac{S_{kl}^2}{(s_l - s_k)(\mathbf{Ai}'(-s_l))^2} \quad (13)$$

where

$$S_{kl} = \int_{-\infty}^{\infty} \mathbf{Ai}(x - s_k) \mathbf{Ai}'(x - s_l) dx,$$

$$\mathbf{Ai}'(x) \equiv \frac{d}{dx} \mathbf{Ai}(x),$$

and $-s_k$ and $-s_l$ are roots of the Airy function. Numerical studies of the function $\mu_k$ have shown its poor dependence on k, so that one can assume it to be $\mu_k \cong \mu = 0{,}249$ with $k \in [1;5]$ and suppose $M_k \cong M$.

Now note that the following condition is normally satisfied for the considered design

$$\lambda_n << b << \sqrt{RL}. \quad (14)$$

We have three types of frequencies, which highly differ for the small parameter $\lambda_n / b << 1$. It turns out that for the fixed values of indexes $n$ and $k$ the quasi one-dimensional spectrum $\omega_{nkl}$ occurs for the index $l$ near the fundamental frequency. Furthermore, if to use in (10) the asymptotic for roots of the Airy function with $k$ >>1, this is

$$s_k \cong \left( \frac{3\pi}{2} \left( k - \frac{1}{4} \right) \right)^{2/3}, \quad (15)$$

then the following approximate relation may be derived from (10) for the resonator frequency spectrum

$$\omega_{nkl}^2 \cong \omega_n^2 \left\{ 1 + \frac{b^2}{RL} \left[ \frac{1}{\gamma_0^2} + \left( \frac{\lambda_n}{b} \right)^{4/3} \beta \left( \frac{3\pi}{2} \right)^{2/3} \left( k - \frac{1}{4} \right)^{2/3} + 2 \left( \frac{\lambda_n}{b} \right)^2 \sqrt{BM} \left( l - \frac{1}{2} \right) \right] \right\}, \quad (16)$$

where all the constants are determined via the crystal resonator design and physical constants.

Let us now discuss the achieved result paying attention to the special features of the frequency spectrum and to the electrode influence. First, let us select the point $C$ just at the closest distance between the electrodes (Fig. 1 and 3*a*). It turns out that the shape of the potential of the equation (6) around the point $C$ influence the spectrum (10). To illustrate, Figure 3*b* exhibits the potential shape, which, of course, varies in a different way for the cut $\xi$ = const (Fig. 3*c*) and $\eta$ = const (Fig. 3*d*).

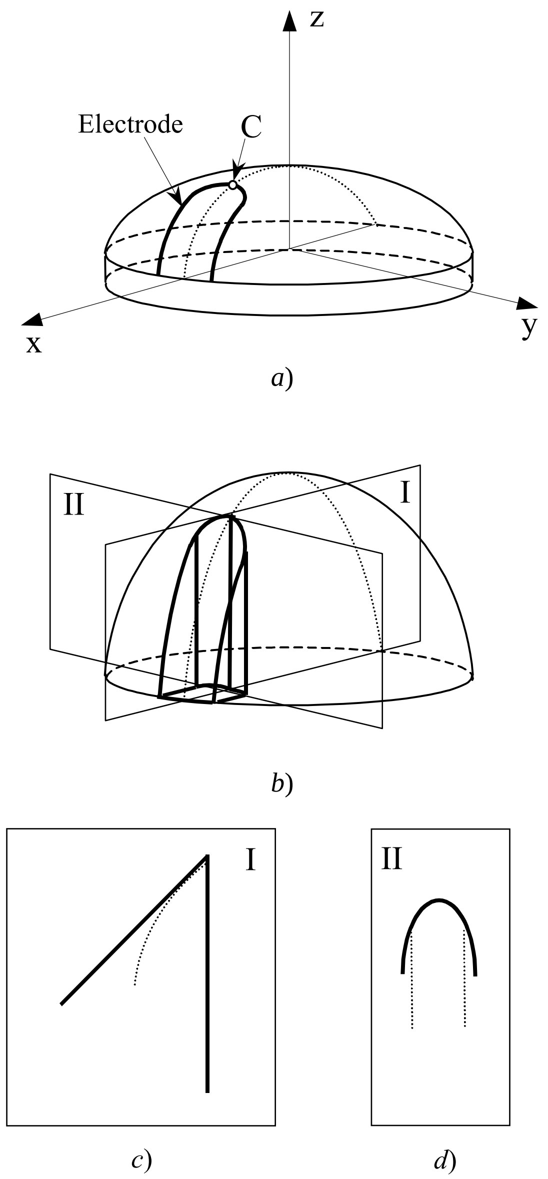


**Figure 3. Potential of the differential equation (6) around the point C, which determines frequencies of self vibrations**

Quasi one-dimensional frequency spectrum occurred in the considered case inevitably contributes for the resonator performance. To account it and design the electrodes in an optimum way for the crystal resonator performance, we plan to carry out additional studies.

## 3. CONCLUSION

An analysis of the frequency spectrum of the convex piezoelectric plate with one side placed electrodes employing thickness-shear vibrations is provided. It is shown that such an analysis is valid for the electrodes of an arbitrary shape, so long as frequencies are determined by the local characteristics of the electrode bound. These characteristics are: a minimum distance *b* between the boundary point *C* and a center of the piezoelectric plate, and a radius *a* of the bound curvature at this point (Fig. 2). As a practical result of the studies, we present the approximate nonlinear relation (16), which, however, sometimes requires to be completed with additional terms. We plan to study this problem in further.

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